A Note on the Optimality of Index Priority Rules for Search and Sequencing Problems

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Abstract: We show that the linear objective function of a search problem can be generalized to a power function and/or a logarithmic function and still be minimized by an index priority rule. We prove our result by solving the differential equation resulting from the required invariance condition, therefore, we also prove that any other generalization of this linear objective function will not lead to an index priority rule. We also demonstrate the full equivalence between two related search problems in the sense that a solution to either one can be used to solve the other one and vice versa. Finally, we show that the linear function is the only function leading to an index priority rule for the single-machine makespan minimization problem with deteriorating jobs and an additive job deterioration function. Naval Research Logistics 58: 83–87, 2011

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1. INTRODUCTION

The operations research literature is concerned with the identification of index priority rules to solve search problems and/or single-machine sequencing problems. For example, consider the following objective function [5]

$$V = \sum_{i=1}^{n} c_i \prod_{j=1}^{i-1} p_j = \sum_{i=1}^{n} c_i p_i^{-1} \prod_{j=1}^{i} p_j, \tag{1}$$

stemming from a search problem with n sequentially positioned boxes; box j contains an object with probability $1-p_j$ and c_j denotes the cost of searching and discovering the object hidden in box j. The objective is to determine the optimal positioning of the boxes so that V is minimized. If the events associated with the probabilities $p_1, \ldots, p_n$ are assumed to be disjoint rather than independent, then Eq. (1) can be written as

$$X = \sum_{i=1}^{n} q_i \sum_{j=1}^{i} d_j, \tag{2}$$

where d_j denotes the search cost and q_j denotes the probability of searching box j. An alternative application of Eq. (2) can be found in the single-machine scheduling literature if q_j is not restricted to the [0,1] interval; in that case, q_j denotes the weight of job j and d_j the completion time of job j in a standard non-preemptive single-machine sequencing problem with constant job processing times and a continuously available single machine which can process at most one job at a time. Both Eqs. (1), (2) can be minimized in $O(n \log n)$ time by implementing an index priority rule according to which each box (job) is assigned a priority computed based exclusively on the characteristics of the box (job); this index is independent of the presence (absence) of other boxes (jobs) in the sequence.

These findings motivated two important research questions:

> Question (a): can Eqs. (1) and (2) be generalized and still be solvable by an index priority rule?
> Question (b): can the similarity between Eqs. (1) and (2) be exploited so that a solution to Eq. (1) can be used to solve Eq. (2) and vice versa?

Rothkopf and Smith [9] answered question (a) in the context of Eq. (2) and concluded that the only possible generalization of Eq. (2) is to a class of exponential functions. They arrived at this conclusion by first stating the required invariance condition and then deriving and solving the resulting differential equation. Rothblum and Rothkopf [8] proved

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that the findings of Rothkopf and Smith [9] hold in a dynamic environment as well.

Kelly [5] answered question (b) and concluded that any schedule σ_0 that strictly minimizes X in (2) (that is, $X(\sigma_0) < X(\sigma)$ for all σ) also strictly minimizes V in (1) under certain conditions linking the expressions in (1) and (2). Kelly [5] reached his conclusion by first linking Eqs. (1) and (2) via a real-valued variable x and then requiring x to assume sufficiently small positive values. We also mention that, in addition to models (1) and (2), Kelly [5] considered the third, "minimax" model proposed by Monma and Sidney [6].

The first objective of this note is to extend the answers to questions (a) and (b). We extend the answer to question (a) by showing that Eq. (1) can be generalized to any power function and/or to a logarithmic function and still be solvable by an index priority rule.

We also extend the answer to question (b) by first linking Eqs. (1) and (2) without depending on any real-valued variable x and then showing the full equivalence of Eqs. (1) and (2) in the sense that a solution to Eq. (1) can be used to solve Eq. (2) and vice versa.

Question (a) has also been posed by Browne and Yechiali [2] in the context of a single-machine sequencing problem with job deterioration due to waiting. In that case, the actual job processing times depend on the job starting times and in the case of a linear job deterioration function can be expressed as

$$Y_i(t) = X_i + \alpha_i t, \tag{3}$$

where $X_i > 0$ is the initial processing requirement of job i, t is the starting time of job i and α_i is the processing growth rate of job i. Browne and Yechiali [2] showed that in the case of the linear job deterioration function (3), the maximum job completion time (makespan) can be minimized by an index priority rule and concluded their paper by stating that it would be of interest to see if any other job deterioration functions yield an index policy.

We answer this question in the case of an additive job deterioration function by first formulating the differential equation resulting from the required invariance condition and then showing that only the linear job deterioration function yields an index policy.

The rest of this note is organized as follows. Section 2 is concerned with the generalization of Eq. (1) so that it is still solvable by an index priority rule. Section 3 is concerned with the equivalence of Eqs. (1) and (2). Section 4 is concerned with the optimality of index priority rules for the single-machine sequencing problem with deteriorating jobs and an additive job deterioration function. Some concluding remarks are stated in Section 5.

2. THE GENERALIZATION OF THE SEARCH PROBLEM

Consider the search problem described by Eq. (1) with n boxes i, $i = 1, \ldots, n$, and suppose that each box i has a general search cost function $C_i(t_i)$, where $t_i = \prod_{j=1}^{i} p_j$. The total cost function is thus given by

$$\sum_{i=1}^{n} C_i(t_i) = \sum_{i=1}^{n} C_i\left(\prod_{j=1}^{i} p_j\right).$$

We are interested in index priority rules for minimizing $\sum_{i=1}^{n} C_i(t_i)$. The well-known index priority rule in the case of the linear function $C_i(t_i) = c_i p_i^{-1} t_i$ in (1) is that $\sum_{i=1}^{n} C_i(t_i)$ is minimized when the tasks are scheduled in the non-decreasing order of the index $\frac{c_i}{1-p_i}$ [7].

As in Rothkopf and Smith [9], we consider a class F of nonnegative, nondecreasing cost functions $C(t)$ closed under positive scalar multiplication and define the following invariance condition which must hold in order for an index priority rule to exist.

2.1. Invariance Condition 1.

For any two boxes i, j with probabilities $0 < p_i < 1$, $0 < p_j < 1$ and cost functions $C_i(t)$ and $C_j(t)$ in F, respectively, the inequality

$$C_i(p_i p_j) - C_i(p_i) \geq C_j(p_i p_j) - C_j(p_j)$$

implies that, for all $\tau \geq 0$,

$$C_i(\tau p_i p_j) - C_i(\tau p_i) \geq C_j(\tau p_i p_j) - C_j(\tau p_j).$$

For an arbitrary cost function $C(t)$ in F and probabilities p_i and p_j for boxes i and j, we define the constant

$$K(p_i, p_j) = \frac{C(p_i p_j) - C(p_i)}{C(p_i p_j) - C(p_j)}. \tag{4}$$

We also define $C_i(t) = C(t)$ and $C_j(t) = K(p_i, p_j)C(t)$, where $C_j(t)$ is in F. The constant $K(p_i, p_j)$ is well defined if the denominator in (4) is nonzero. The proof of Theorem 1 below only requires that $K(p_i, p_j)$ be well defined for the p_j values arbitrarily close to 1, $0 < p_j < 1$. Then, if $C(p_i) < C(1^-)$ for some $0 < p_i < 1$, it follows that $C(p_i p_j) - C(p_j) < 0$ for p_j sufficiently close to 1, $0 < p_j < 1$, provided that $C(t)$ is continuous, thus implying that $K(p_i, p_j)$ is well defined. Equation (4) implies that

$$C(p_i p_j) - C(p_i) = K(p_i, p_j)[C(p_i p_j) - C(p_j)]. \tag{5}$$

Equation (5) in conjunction with the Invariance Condition 1 implies that, for all $0 < p_i < 1$, $0 < p_j < 1$, $\tau \geq 0$,

$$C(\tau p_i p_j) - C(\tau p_i) = K(p_i, p_j)[C(\tau p_i p_j) - C(\tau p_j)]. \tag{6}$$

The next result is analogous to the result in Rothkopf and Smith [9] for cost functions of the form $C_i(t_i)$, where $t_i = \sum_{j=1}^{i} p_j$.

THEOREM 1: Suppose that Eq. (6) holds for all $0 < p_i < 1$, $0 < p_j < 1$ and all $\tau \geq 0$ and that $C(t)$ is twice continuously differentiable everywhere except at most at a finite number of points. Then, $C(t)$ must have one of the following functional forms

$$C(t) = At^{\lambda} + B, \lambda \neq 0, \text{ or } C(t) = A \ln t + B,$$

where λ is the same for all functions $C(t)$.

PROOF: The proof that $C(t)$ is twice differentiable for all $0 < t < 1$ is analogous to that in Rothkopf and Smith [9].

Suppose that $C(p_i) < C(1^-)$ where $0 < p_i < 1$. In view of (4), Eq. (6) can be written as

$$C(\tau p_i p_j) - C(\tau p_i) = \frac{C(p_i p_j) - C(p_i)}{C(p_i p_j) - C(p_j)}[C(\tau p_i p_j) - C(\tau p_j)]. \tag{7}$$

If we divide both sides of (7) by $p_j - 1$ and take the limit as $p_j \to 1$, we obtain

$$\tau p_i C'(\tau p_i) = \frac{p_i C'(p_i)}{C(p_i) - C(1^-)}[C(\tau p_i) - C(\tau)]. \tag{8}$$

After dividing both sides of (8) by p_i, we subtract $C'(p_i)$ from both sides to obtain

$$\tau C'(\tau p_i) - C'(p_i) = C'(p_i)\left[\frac{C(\tau p_i) - C(\tau) - C(p_i) + C(1^-)}{C(p_i) - C(1^-)}\right]. \tag{9}$$

If we divide both sides of (9) by $\tau - 1$ and take the limit as $\tau \to 1$, we obtain

$$p_i C''(p_i) + C'(p_i) = \frac{C'(p_i)}{C(p_i) - C(1^-)}[p_i C'(p_i) - C'(1^-)]. \tag{10}$$

One can verify that the first derivative of the expression

$$p_i C'(p_i)[C(p_i) - C(1^-)]^{-1} - C'(1^-)[C(p_i) - C(1^-)]^{-1} \tag{11}$$

is

$$\begin{aligned} &p_i C''(p_i)[C(p_i) - C(1^-)]^{-1} \\ &\quad - p_i[C'(p_i)]^2[C(p_i) - C(1^-)]^{-2} \\ &\quad + C'(p_i)[C(p_i) - C(1^-)]^{-1} \\ &\quad + C'(p_i)C'(1^-)[C(p_i) - C(1^-)]^{-2}. \end{aligned} \tag{12}$$

In view of (11) and (12), (10) implies that

$$p_i C'(p_i)[C(p_i) - C(1^-)]^{-1} - C'(1^-)[C(p_i) - C(1^-)]^{-1} = D$$

for some constant D, which can be written as

$$tC'(t) - DC(t) = C'(1^-) - DC(1^-), \tag{13}$$

where $0 < t < 1$. This first order linear differential equation has a general solution of the form $C(t) = A \ln t + B$ for $D = 0$. When $D \neq 0$, Eq. (13) has a general solution $C(t) = At^{\lambda} + B$, where $\lambda \neq 0$.

It is of interest to point out that there is a more general family of functions (the power functions) that can be substituted in Eq. (1) and still be optimized by an index priority rule compared to the findings of Rothkopf and Smith [9] for the generalization of Eq. (2). □

3. THE EQUIVALENCE OF THE TWO SEARCH PROBLEMS

In this section, we extend the partial result on the equivalence between the minimization of V [given by Eq. (1)] and the minimization of X [given by Eq. (2)] obtained by Kelly [5] by imposing an additional restrictive assumption that $0 < d_i < 1$ for all i which is not present in Kelly [5]. The common features of Eqs. (1) and (2) have also been investigated by Rau [7], Kadane and Simon [4] and Kadane [3].

THEOREM 2: If $q_i = -c_i$ and $d_i = 1 - p_i$, where $0 < p_i < 1$, $0 < d_i < 1$, then the minimization of V is equivalent to the minimization of X.

PROOF: Observe that the expression for V in Eq. (1) is minimized when the tasks are scheduled in the non-decreasing order of the index $\frac{c_i}{1-p_i}$ [7]. Similarly, observe that the expression for X in Eq. (2) is minimized when the tasks are scheduled in the non-decreasing order of the index $\frac{d_i}{q_i}$ [7]. Since $q_i = -c_i$ and $d_i = 1 - p_i$, X is minimized when the tasks are scheduled in the non-decreasing order of the index $\frac{1-p_i}{-c_i}$ which is equivalent to the non-decreasing order of the index $\frac{c_i}{1-p_i}$.

Kelly [5] states that one can obtain the solution that minimizes X by finding the solution that minimizes V. Observe that Kelly [5] requires $x > 0$ to be small so that $p_i = 1 - xd_i$ is positive regardless of the d_i value. Under the additional assumption that $0 < d_i < 1$ for all i, we show that one can also obtain the solution that minimizes V by finding the solution that minimizes X. As a result, we can eliminate the use of the additional variable x when relating the constants in V and X unlike Kelly [5] who considers a general case of d_i values. Observe also that both in Theorem 2 and in Theorem 1 in Kelly [5] the quantities c_i and q_i cannot be both positive. □

4. THE OPTIMALITY OF INDEX PRIORITY RULES WITH DETERIORATING JOBS

In most scheduling problems, it is assumed that the job processing times are constant. Browne and Yechiali [2] were among the first to consider job deterioration due to waiting, resulting in the actual job processing times given by Eq. (3). Browne and Yechiali [2] showed that the makespan can be minimized by an index priority rule when the actual job processing times are given by Eq. (3); they also stated two related open problems.

Problem (P1): What is the complexity status of the weighted job completion time problem when linear job deterioration is in effect?

Ref. 2 conjectured that the problem is NP-hard; subsequently, [1] proved that the problem is indeed NP-hard.

Problem (P2): Does an index priority rule exist for the makespan minimization problem when the degree of job deterioration can no longer be assumed proportional to the job waiting time? For example, in the case of an additive job deterioration function, does an index priority rule exist when the actual processing time $Y_i(t)$ of job i is given as

$$Y_i(t) = X_i + f_i(t), \quad f_i(0) = 0, \; i = 1, \ldots, n, \tag{14}$$

where $f_i(t)$ is a function of the starting time t of job i?

Our objective is to solve problem (P2) when Eq. (14) are in effect. We first derive the required invariance condition.

Denote by S_k and Y_k the actual completion time and the actual processing time of the kth job in the sequence, respectively, $k = 1, \ldots, n$. Then, $Y_1 = Y_1(S_0) = X_1$, where $S_0 = 0$; $S_k = \sum_{i=1}^{k} Y_i$, where $Y_i = X_i + f_i(S_{i-1})$, $i = 1, \ldots, k$, $k = 1, \ldots, n$. The makespan is defined as S_n.

We consider a class G of nonnegative, nondecreasing cost functions $f(t)$ (where $f(0) = 0$) closed under positive scalar multiplication and define the following invariance condition which must hold for the existence of an index priority rule to minimize S_n when Eqs. (14) are in effect.

4.1. Invariance Condition 2.

For any two tasks i, j with $X_i > 0$ and $X_j > 0$ and cost functions $f_i(t)$ and $f_j(t)$ in G, respectively, the inequality

$$f_i(X_j) \geq f_j(X_i)$$

implies that, for all $x \geq 0$,

$$f_i(x + X_j + f_j(x)) - f_i(x) \geq f_j(x + X_i + f_i(x)) - f_j(x)$$

(the above inequality is well defined because $f_j(x) \geq 0$, $f_i(x) \geq 0$ for $x \geq 0$.)

For an arbitrary cost function $f(t)$ in G and X_i and X_j for jobs i and j, we define the constant

$$M(X_i, X_j) = \frac{f(X_j)}{f(X_i)}. \tag{15}$$

We also define $f_i(t) = f(t)$ and $f_j(t) = M(X_i, X_j) f(t)$, where $f_j(t)$ is in G. The constant $M(X_i, X_j)$ is well defined for $X_i > 0$, $X_j > 0$ if $f(t) > 0$ for any $t > 0$. Equation (15) implies that

$$f(X_j) = M(X_i, X_j) f(X_i). \tag{16}$$

Equation (16) in conjunction with the Invariance Condition 2 implies that, for all $X_i > 0$, $X_j > 0$, $x \geq 0$,

$$\begin{aligned} &f(x + X_j + M(X_i, X_j) f(x)) - f(x) \\ &\quad = M(X_i, X_j)[f(x + X_i + f(x)) - f(x)]. \end{aligned} \tag{17}$$

THEOREM 3: Suppose that Eq. (17) holds for all $X_i > 0$, $X_j > 0$ and all $x \geq 0$ and that $f(t)$ is twice continuously differentiable for $t \geq 0$ and $f(0) = 0$. Then, $f(t)$ must have the following functional form

$$f(t) = At.$$

PROOF: Suppose that $f(t) > 0$ for any $t > 0$. In view of (15), Eq. (17) can be written as

$$\begin{aligned} &f\left(x + X_j + \frac{f(X_j)}{f(X_i)} f(x)\right) - f(x) \\ &\quad = \frac{f(X_j)}{f(X_i)}[f(x + X_i + f(x)) - f(x)]. \end{aligned} \tag{18}$$

If we divide both sides of (18) by X_j and take the limit as $X_j \to 0$, we obtain

$$\begin{aligned} &f'(x)\left[1 + f'(0^+)\frac{f(x)}{f(X_i)}\right] \\ &\quad = \frac{f'(0^+)}{f(X_i)}[f(x + X_i + f(x)) - f(x)]. \end{aligned} \tag{19}$$

After multiplying both sides of (19) by $f(X_i)$, we subtract $f'(0^+)[f(X_i) + f'(0^+) f(x)]$ from both sides to obtain

$$\begin{aligned} &[f'(x) - f'(0^+)][f(X_i) + f'(0^+) f(x)] \\ &\quad = f'(0^+)[f(x + X_i + f(x)) - f(X_i)] \\ &\qquad - f'(0^+) f(x)[1 + f'(0^+)]. \end{aligned} \tag{20}$$

If we divide both sides of (20) by x and take the limit as $x \to 0$, we obtain

$$f''(0^+)f(X_i) = f'(0^+)[1 + f'(0^+)][f'(X_i) - f'(0^+)]. \tag{21}$$

By dividing both sides of (21) by X_i and taking the limit as $X_i \to 0$, we obtain

$$f''(0^+)f'(0^+) = f'(0^+)[1 + f'(0^+)]f''(0^+)$$

which implies that $f''(0^+)[f'(0^+)]^2 = 0$. If $f'(0^+) \neq 0$ then $f''(0^+) = 0$ which, in view of Eq. (21), implies that $f'(X_i) = f'(0^+)$. This in turn implies that $f(t) = At$ since $f(0) = 0$. On the other hand, if $f'(0^+) = 0$ then, in view of Eq. (19), $f'(x) = 0$ for any $x \geq 0$. This implies that $f(t) = 0$ since $f(0) = 0$.

Consequently, problem (P2) can be solved by an index priority rule only when the general additive Eq. (14) reduce to the linear Eq. (3). □

5. CONCLUDING REMARKS

We showed that the expression in Eq. (1) can be generalized to a power function and/or a logarithmic function and still be minimized by an index priority rule. We proved our result by solving the differential equation resulting from the required invariance condition, therefore, our proof implies that any other generalization of Eq. (1) will not lead to an index priority rule. Our findings indicate that there is a more general family of functions (the power functions) that can be substituted in Eq. (1) and still be optimized by an index priority rule compared to the similar findings of Rothkopf and Smith [9] for the similar Eq. (2).

We also demonstrated the full equivalence between the two related search problems represented by Eqs. (1) and (2) in the sense that a solution to either one can be used to solve the other one as well. This finding strengthens a previous finding of Kelly [5] on this issue.

In the case of the single-machine sequencing problem with job deterioration, we showed that the linear function is the only function leading to an index priority rule for the makespan minimization problem when an additive job deterioration function is in effect.

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