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Short Communication

A note on single-machine scheduling with job-dependent learning effects

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ABSTRACT

Mosheiov and Sidney (2003) showed that the makespan minimization problem with job-dependent learning effects can be formulated as an assignment problem and solved in $O(n^3)$ time. We show that this problem can be solved in $O(n\log n)$ time by sequencing the jobs according to the shortest processing time (SPT) order if we utilize the observation that the job-dependent learning rates are correlated with the level of sophistication of the jobs and assume that these rates are bounded from below. The optimality of the SPT sequence is also preserved when the job-dependent learning rates are inversely correlated with the level of sophistication of the jobs and bounded from above.

1. Introduction

Biskup (1999) and Mosheiov (2001) showed that, in a learning environment, a number of single-machine scheduling problems can be formulated as assignment problems and then solved in $O(n^3)$ time. In these problems, the learning phenomenon is incorporated by defining the actual processing time of job j when scheduled in position r in the sequence, $p_{j[r]}$, as

$$p_{j[r]} = p_j r^{\alpha}, \tag{1}$$

where $\alpha < 0$ is the applicable learning rate and p_j is the "nominal" processing time of job j, $j = 1,\ldots,n$. It was shown later (Biskup and Simmons, 2004) that some of these problems can be solved in $O(n\log n)$ time using a weight-matching approach which utilizes the observation that the learning effect (the r^{α} term) can be incorporated into the "positional weight" (the contribution) of the job to the objective function.

Mosheiov and Sidney (2003) generalized Biskup's (1999) learning model to incorporate job-dependent learning effects. In that case, Eq. (1) is replaced by the more general equation

$$p_{j[r]} = p_j r^{\alpha_j}, \tag{2}$$

where $\alpha_j < 0$ denotes the job-dependent learning rate for job j, $j = 1,\ldots,n$. Mosheiov and Sidney (2003) argued that this more general model is justified because the learning process of a worker may be significantly affected by the job itself.

It was shown by Mosheiov and Sidney (2003) that the assignment formulations of Biskup (1999) and Mosheiov (2001) can be used to solve a series of single-machine scheduling problems in $O(n^3)$ time when Eq. (2) are in effect. However, the $O(n\log n)$ weight-matching approach cannot be used when Eq. (2) are in effect because the learning effect (the r^{α_j} term) is job-dependent and its incorporation into the job's positional weight will make it job-dependent as well.

The purpose of this note is to show that the makespan minimization problem with job-dependent learning effects can be solved in $O(n\log n)$ time by utilizing the observation that the job-dependent learning rates are correlated with the level of sophistication of the jobs and by assuming that these rates are bounded from below. We also prove the optimality of the SPT sequence when the job-dependent learning rates are inversely correlated with the level of sophistication of the jobs and bounded from above.

2. The makespan minimization problem with job-dependent learning effects

We consider the standard non-preemptive single-machine makespan minimization problem with job-dependent learning effects given by (2), to be called the $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem from now on. Let C_j denote the completion time of job j, $j = 1,\ldots,n$. The objective is to determine an optimal sequence S^* which minimizes the makespan $C_{\max} = \max_{j=1,\ldots,n}\{C_j\}$.

It has been observed in practical applications that the job-dependent learning rates are correlated with the level of sophistication of the jobs in the sense that there is more room for improving the processing of the more highly sophisticated jobs due to learning compared to simpler jobs. This observation can be quantified by assuming that the more highly sophisticated (longer) jobs experience steeper learning curves compared to the simpler (shorter) jobs. Equivalently, this observation can be stated as follows:

$$\text{if } p_i \leqslant p_j, \text{ then } \alpha_i \geqslant \alpha_j \quad \text{for all } i,j = 1,\ldots,n. \tag{3}$$

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The consideration of (3) facilitates the solution of the $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem in $O(n\log n)$ time by sequencing the jobs according to the shortest processing time (SPT) order $p_1 \leqslant \cdots \leqslant p_n$ (with $\alpha_i \geqslant \alpha_{i+1}$ whenever $p_i = p_{i+1}$, $i = 1,\ldots,n-1$). Since the proof of the optimality of the SPT sequence for the $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem is based on an adjacent job interchange argument, we also need to take into account the findings of the following Lemma.

Lemma 1. *Let $f(x) = r^x - (r+1)^x$, $r = 1,\ldots,n-1$, $x < 0$. Then, $f'(x) \leqslant 0$ when*

$$x \geqslant x^* = \frac{\ln\left(\frac{\ln(r+1)}{\ln(r)}\right)}{\ln(r) - \ln(r+1)} \tag{4}$$

and for all $r = 2,\ldots,n$.

Proof. By computing the derivative $f'(x)$ and then solving the inequality $f'(x) \leqslant 0$; selective pairs of $\langle r, x^* \rangle$ values can be computed from expression (4) as $\langle 2,-1.14\rangle$, $\langle 5,-0.59\rangle$, $\langle 10,-0.43\rangle$, $\langle 20,-0.33\rangle$, $\langle 50,-0.25\rangle$ and $\langle 100,-0.22\rangle$ respectively. □

We now state the main result of this note.

Proposition 1. *The $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem can be solved in $O(n\log n)$ time by sequencing the jobs according to the SPT order $p_1 \leqslant \cdots \leqslant p_n$ (with $\alpha_i \geqslant \alpha_{i+1}$ whenever $p_i = p_{i+1}$, $i = 1,\ldots,n-1$) when conditions (3) are in effect and also*

$$\min_{j=1,\ldots,n}\{\alpha_j\} \geqslant \frac{\ln\left(\frac{\ln(n)}{\ln(n-1)}\right)}{\ln(n-1) - \ln(n)}. \tag{5}$$

Proof. By a standard adjacent job interchange argument. Assume that there is an optimal sequence with jobs j, i sequenced in positions r and $r+1$ respectively where $r \in \{1,\ldots,n-1\}$ and either $p_j > p_i$ or $p_j = p_i$ and $\alpha_j < \alpha_i$. Now consider the job sequence after interchanging the positions of jobs j, i. The contribution to $C_{\max}$ of all jobs sequenced before and after jobs j, i is not affected by this job interchange. The combined contribution of jobs i, j to $C_{\max}$ before and after the i, j job interchange is $p_j r^{\alpha_j} + p_i(r+1)^{\alpha_i}$ and $p_i r^{\alpha_i} + p_j(r+1)^{\alpha_j}$ respectively. It suffices to show that $p_i r^{\alpha_i} + p_j(r+1)^{\alpha_j} < p_j r^{\alpha_j} + p_i(r+1)^{\alpha_i}$. After rearranging terms, the above inequality can be written as

$$p_i[r^{\alpha_i} - (r+1)^{\alpha_i}] < p_j[r^{\alpha_j} - (r+1)^{\alpha_j}]. \tag{6}$$

By Lemma 1, the condition (5) ensures that the function $f(r) = r^{\alpha} - (r+1)^{\alpha}$ is a decreasing function for all $r = 1,\ldots,n-1$ and for all α_j, $j = 1,\ldots,n$. This observation together with the conditions (3) ensures the validity of the inequality (6) because all four product terms in (6) are positive. Consequently, the i, j job interchange does not increase the $C_{\max}$ value. The repeated application of the above argument leads to the optimality of the SPT sequence for the $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem.

The arguments of the proof of Proposition 1 can also be used to prove the optimality of the SPT sequence when the job-dependent learning rates are inversely correlated with the level of sophistication of the jobs and bounded from above. In that case,

$$\text{if } p_i \leqslant p_j, \text{ then } \alpha_i \leqslant \alpha_j \quad \text{for all } i,j = 1,\ldots,n \tag{7}$$

and also

$$\max_{j=1,\ldots,n}\{\alpha_j\} \leqslant \frac{\ln\left(\frac{\ln(n)}{\ln(n-1)}\right)}{\ln(n-1) - \ln(n)}. \tag{8}$$

The conditions (7) and (8) ensure the validity of the inequality (6) leading to the following corollary to Proposition 1 which is stated next without proof; recall, that in this case, the function $f(x) = r^x - (r+1)^x$ is an increasing function. □

Corollary 1. *The $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem can be solved in $O(n\log n)$ time by sequencing the jobs according to the SPT order $p_1 \leqslant \cdots \leqslant p_n$ (with $\alpha_i \leqslant \alpha_{i+1}$ whenever $p_i = p_{i+1}$, $i = 1,\ldots,n-1$) when conditions (7) and (8) are in effect.*

3. Discussion and conclusions

The $1/p_{j[r]} = p_j r^{\alpha_j}/C_{\max}$ problem can be solved in $O(n\log n)$ time by sequencing the jobs according to the SPT order if we utilize the observation that the job-dependent learning rates are correlated with the level of sophistication of the jobs and assume that these rates are bounded from below. The optimality of the SPT sequence is also preserved when the job-dependent learning rates are inversely correlated with the level of sophistication of the jobs and bounded from above. Our approach demonstrates the robustness of the SPT sequence which is also the optimal sequence for the $1/p_{j[r]} = p_j r^{\alpha}/C_{\max}$ problem with job-independent learning rates.

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